APPLICATION OF GLOBAL NUMERICAL PROCEDURES TO THE ANALYSIS OF SHELLS

Avelino SAMARTIN
Universidad Politecnica de Madrid
Department of Structural Mechanics
E.T.S.I de Caminos, Canales y Puertos
28040 Madrid, Spain
E-mail: samartin@caminos.upm.es

Abstract

Examples of global solutions of the shell equations are presented, such as the ones based on the well known Levy series expansion. Also discussed are some natural extensions of the Levy method as well as the inherent limitations of these methods concerning the shell model assumptions, boundary conditions and geometric regularity. Finally, some open additional design questions are noted mainly related to the simultaneous use in analysis of these global techniques and the local methods (like the finite elements) to finding the optimal shell shape, and to determining the reinforcement layout.

1. Introduction

The shallow curved plate theory of linear elastic thin shells is assumed. Details can be seen in [1]. A right hand cartesian axis (x_1, x_2, z) is used to describe the middle shell surface by means the following parametric equations:

$$x_1 = x_1(\alpha_1, \alpha_2); x_2 = x_2(\alpha_1, \alpha_2); z = z(\alpha_1, \alpha_2)$$

The following notation is used:

- Indexes i and j vary between 1 and 2 ($i \neq j$).
- δ_{ij} is the Kronecker delta.
- Einstein summation convention applies unless the contrary is explicitly stated.
- Comma notation with index i is used to represent partial derivatives respect to α_i.
- X_i, Z are the pressure force components on the middle surface.
- u_i, w are the displacement components at a point of the middle surface.
- n_{ij} and q_i are the force stress-resultants of the stresses σ_{ij} and σ_{iz}.
- m_{ij} are the couple stress-resultants of the stresses σ_{ij}.
- A_{ij} are the coefficients of the first fundamental form of the middle surface. It is assumed $A_{ij} = 0$.
- K_{ij} are the curvatures of the middle surface.
- h is the constant shell thickness, E the Youngs modulus, ν the Poisson ratio, $D = \frac{Eh^3}{12(1-\nu^2)}$ and $K = \frac{Eh}{(1-\nu^2)}$.

The equilibrium equations of a differential shell element are:

$$\begin{aligned} n_{ij,j} + X_i &= 0 \\ m_{ij,j} - q_i &= 0 \\ K_{ij} n_{ij} + q_{i,j} + Z &= 0 \end{aligned}$$

The following main relations between the different shell variables hold:

- Strains/ displacements

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} - 2K_{ij}w)$$

$$k_{ij} = -w_{,ij}$$

- Stress-resultants/strains

$$n_{ij} = K[(1-\nu)e_{ij} + \nu\delta_{ij}e_{rr}$$

$$m_{ij} = D[(1-\nu)k_{ij} + \nu\delta_{ij}k_{rr}]$$

- Stress-resultants/ displacements

$$n_{ii} = K[u_{i,i} + \nu u_{j,j} - (K_{ii} + \nu K_j j)w]$$

$$n_{ij} = K[u_{i,j} + u_{j,i} - 2K_{ij}w]$$

$$q_i = -D\nabla^2 w_{,j}$$

$$r_i = q_i + m_{ij,j} = -D[w_{,iii} + (2-\nu)w_{,jjj}]$$

in which r_i are the Kirchhoff shears and $\nabla^2 = \frac{\partial^2}{\partial\alpha_1^2} + \frac{\partial^2}{\partial\alpha_2^2}$.

The governing differential equations expressed in terms of the normal displacement w and the Pücher stress function Φ, defined by the expression (not summed) $n_{ij} = (-)^{i+j}\Phi_{ij} - \delta_{ij}\int X_i d\alpha_i$, are:

$$D\nabla^4 w - \nabla_K^2\Phi = Z - K_{ii}\int X_i d\alpha_i$$

$$\nabla^4\Phi - Eh\nabla_K^2 w = \int X_{i,jj} d\alpha_i + \nu X_{i,i}$$

in which $\nabla_K^2 = K_{ii}\frac{\partial^2}{\partial\alpha_j^2} - K_{ij}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}$.

In the case of $\nabla_K^2 \neq \nabla^2$ the Ambartsuyam function W is normally introduced [3] to obtain the complementary solution by solving a single differential equation:

$$\nabla^8 W + \frac{12(1-\nu^2)}{h^2}\nabla_K^4 W = 0$$

in which $w = \nabla^4 W; \Phi = -Eh\nabla_K^2 W$.

In the other case ($K_{11} = K_{22}; K_{12} = 0$ i.e. spherical shallow curved plate) the Mishonov function is used instead, defined as: $\nabla^2 w = \nabla^2 W; \nabla^2\Phi = -EhK\nabla_K^2 W$, and the same single differential equation as before is reached except the coefficient of second term is now multiplied by K.

2. Description of the Levy solution.

Rectangular planform, curvature lines ($K_{12} = 0$) and normal gable boundary conditions along two opposite edges of the shell are assumed, i.e.:

$$n_{11} = 0; u_2 = 0; w = 0; m_{11} = 0 \text{ along } \alpha_1 = 0, L_1$$

For brevity only the case with different curvatures is shown. The spherical solution $K_{11} = K_1 = K_{22} = K_2$ follows a similar pattern.

The boundary conditions along the two other edges $\alpha_2 = 0, L_2$ can be quite arbitrary.

The solution is expressed by sum of harmonic terms. For each term a vector R of dimension 15x1 containing the results of interest in the analysis is defined as follows:

$$R = (u_1, u_2, w, w_{,1}, w_{,2}, n_{11}, n_{22}, n_{12}, m_{11}, m_{22}, m_{12}, q_1, q_2, r_1, r_2)^T$$

in which each element is a function of α_2 and varies along the direction α_1 as $\sin \lambda\alpha_1$ except the terms 1, 4, 8, 11, 12 and 14 which vary as $\cos \lambda\alpha_1$, $\lambda = n\frac{\pi}{L_1}$ corresponds to the n-th expansion term. The expression of this vector of results is given as sum of a particular and the complementary solutions [2]:

$$R = R_0 + R_c = R_0 + G[C_1 P(\alpha_2) A + C_2 P(\beta_2) B]$$

in which G is a 15×8 matrix shown in table 1. $C_i = [C_1^i, C_2^i]$ is a partitioned matrix of dimension 8×4 and the k-th row of the 8×2 submatrix C_j^i is $\epsilon_{jk}[\rho_i^k \cos k\varphi_i, \rho_i^k \sin k\varphi_i]$ with $\epsilon_{1k} = (-1)^k, \epsilon_{2k} = 1; k = 0, 1, ..., 7, \rho_i = (r_i^2 + s_i^2)^{\frac{1}{2}}$ and r_i and s_i are constants depending on the real and imaginary parts of the roots of the characteristic equation,:

$$r_i = \left[\frac{a_i + \sqrt{a_i^2 + b_i^2}}{2}\right]^{\frac{1}{2}}; s_i = \left[\frac{-a_i + \sqrt{a_i^2 + b_i^2}}{2}\right]^{\frac{1}{2}}$$

$$a_i = \lambda^2 - (-1)^i \mu \sqrt{\frac{-K_1^2 + \Delta}{2}}; b_i = \mu \left[\frac{K_1^2 + \Delta}{2} + (-1)^i K_1 \frac{K_2 - K_1}{|K_2 - K_1|}\right]$$

$$\mu = \frac{\sqrt{3(1-\nu^2)}}{h}; \qquad \Delta = \sqrt{K_1^4 + 4(K_1 - K_2)^2 \frac{\lambda^4}{\mu^2}}$$

The square matrix $P(x) = [P_{ij}(x)], x = \alpha_2, \beta_2 = L_2 - \alpha_2$ of dimension 4 is partitioned in 4 submatrices of dimension 2×2. Each of these submatrices has the following expression:

$$P_{ii}(x) = \begin{bmatrix} p_{i1}(x) & p_{i2}(x) \\ -p_{i2}(x) & p_{i1}(x) \end{bmatrix}; P_{ij}(x) = 0; \quad p_{i1} = e^{-r_i x} \cos s_i x; \quad p_{i2} = e^{-r_i x} \sin s_i x$$

The eight arbitrary constants A_{ij}, B_{ij} are contained in the two 4×1 column matrices

$$A = (A_{11}, A_{12}, A_{21}, A_{22})^T; B = (B_{11}, B_{12}, B_{21}, B_{22})^T$$

These constants can be found by imposing the arbitrary boundary conditions along the edges $\alpha_2 = 0, L_2$

TABLE 1. Matrix $G = [G_1, G_2]$

$$G_1 = \begin{bmatrix} (-K_1 + \nu K_2)\lambda^3 & 0 & -K_2\lambda + (2+\nu)K_1\lambda & 0 \\ 0 & K_1\lambda^2 - (2+\nu)K_2\lambda^2 & 0 & K_2 + \nu K_1 \\ \lambda^4 & 0 & -2\lambda^2 & 0 \\ \lambda^5 & 0 & -2\lambda^3 & 0 \\ 0 & \lambda^4 & 0 & -2\lambda^2 \\ 0 & 0 & K_2^2\lambda^2 Eh & 0 \\ -K_2\lambda^4 Eh & 0 & K_1\lambda Eh & 0 \\ 0 & -K_2\lambda^3 Eh & 0 & K_1\lambda Eh \\ \lambda^6 & 0 & -(2+\nu)\lambda^4 D & 0 \\ \nu\lambda^6 D & 0 & -(1+2\nu)\lambda^4 D & 0 \\ 0 & -(1+\nu)\lambda^5 D & 0 & 2(1-\nu)\lambda^3 D \\ \lambda^7 D & 0 & -3\lambda^5 D & 0 \\ 0 & \lambda^6 D & 0 & -3\lambda^4 D \\ \lambda^7 D & 0 & -(4-\nu)\lambda^5 D & 0 \\ 0 & (2-\nu)\lambda^6 D & 0 & -(5-2\nu)\lambda^4 D \end{bmatrix}$$

$$
G_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\lambda & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-K_1 Eh & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
(1+2\nu)\lambda^2 D & 0 & -\nu D & 0 \\
(2+\nu)\lambda^2 D & 0 & -D & 0 \\
0 & -(1-\nu)\lambda D & 0 & 0 \\
3\lambda^3 D & 0 & -\lambda D & 0 \\
0 & 3\lambda^2 D & 0 & -D \\
(5-2\nu)\lambda^3 D & 0 & -(2-\nu)\lambda D & 0 \\
0 & (4-\nu)\lambda^2 D & 0 & -D
\end{bmatrix}
$$

3. Extensions of the Levy process.

The Levy formulation just described can be extended [2] to treat quite general systems of shallow shell structures, with transversally variable thickness or curvature changes, by introducing the standard methods of matrix analysis of structures, such as the stiffness method, along the transversal shell direction, i.e. the α_2 direction. By a suitable selection of columns and rows of the matrix G and the vector R_0 the two following relations can be reached:

$$
\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = d_0 + G_d \begin{bmatrix} A \\ B \end{bmatrix}; \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = p_0 + G_p \begin{bmatrix} A \\ B \end{bmatrix}
$$

in which p_i and d_i are the vectors of the forces and displacements along the border $\alpha_2 = 0$ and $\alpha_2 = L_2$ respectively, i.e., the vectors with components $n_{12}, n_{22}, r_1, m_{22}$ and $u_1, u_2, w, w_{,2}$. By elimination of the constants A and B the stiffness fundamental equation is obtained:

$$
\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = p_0 - k d_0 + k \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}
$$

with $k = G_p G_d^{-1}$ the 8×8 stiffness matrix.

To treat other boundary conditions than the normal gables in the two opposite edges $\alpha_1 = 0, L_1$ several attempts have been made. The first introduce further approximations in the theory [4], in such a way that the derivatives of order 6 and 2 in the direction α_1 disappear. Some typical approximations are summarized in table 2. In these cases the trigonometric functions $\sin \lambda\alpha_1$ and $\cos \lambda\alpha_1$ used in the Levy process can be replaced by the new orthogonal set of functions in $[0, L_1]$, known as Raleigh functions, $\phi(\alpha_1)$. These functions satisfy more general boundary conditions than the normal gable ones along the two opposite shell edges and they are defined by the eigenvalue problem: $\phi_{,1111} + \lambda^4 \phi = 0$ and the general homogenous boundary conditions. In the formulation of the equations of table 2, the static-kinematic analogy of Goldenweizer [5] has been considered.

Table 2. Approximations of the shell equations.

Approximations		General Plate			Shallow Curved Plate		
State	Functions neglected	Name	Order derivatives α_1	Order derivatives α_2	Name	Order derivatives α_1	Order derivatives α_2
1	None	-	8,6,4,2	8,6,4,2	Donell Jenkins	8,6,4,2	8,6,4,2
2	m_{12}, ϵ_{12}	-	8,4	8,6,4,2	-	8,4	8,4
3	m_{12}, ϵ_{12} m_{11}, ϵ_{22}	Vlasov tvb	4	8,6,4	Schorer	4	8

Another possibility to extend the Levy process has been investigated by Gunasekera [6]. It consists of using three groups of series expansions. The first corresponds to the particular solution and may be a double trigonometric series i.g. the Navier solution. The complementary solution is constructed of two groups of linear independent Levy solutions, along of each the directions α_1 and α_2. The procedure combines suitable sets of Levy solutions with the particular solution to satisfy all the boundary conditions simultaneously. Good convergence has been reported mainly for kinematic boundary conditions.

Finally Michael [7] has developed several strategies to extend the Levy process to more general boundary conditions. The line techniques has been applied to the direct differential equations either in the two variables (normal displacement and stress function) or in the three displacements. Indirect solutions i.e. those using a variational approach or a Raleigh-Ritz method with or without Lagrange multipliers have proven to be efficient even with "difficult" boundary conditions. In table 3 the functions selected in these methods are shown. The suitable selection of these approximating functions is essential for the convergence of the methods.

Table 3.- Approximating functions

Group		Functions
I	A	$\sin m\pi\xi_2$ or $\cos m\pi\xi_2$
	B	$\sin m\pi\xi_2, \cos m\pi\xi_2$
	C	$1, (1-2\xi_2)\sin m\pi\xi_2$
	D	$[\cos n\pi\xi_2 - \cos(n+2)\pi\xi_2]$
	E	$[\cos n\pi\xi_2 - \cos(n+2)\pi\xi_2], [\sin m\pi\xi_2 - \frac{m}{m+2}\cos(m+2)\pi\xi_2]$
II	A	Raleigh functions for the clampled-campled case
	B	$1, (1-2\xi_2), F_m$ where F_m are Raleigh functions for the free-fre case
$m = 1, 2, 3, \cdots n = 0, 1, 2, 3, \cdots \xi_2 = \frac{\alpha_2}{L_2}$		

The functions IA, IB, IC and IIB have been used by Chuang and Veletsos, while Noor and Veletsos have used IB and ID. The functions IA, IIA and IIB are orthogonal functions. The functions ID satisfy the clamped boundary conditions but produce zero normal and Kirchhoff shears on the boundary. They have been proposed in [8] when they are referred as *almost orthogonal functions*. The functions IE have been obtained by modifying ID to avoid the zero shears along the boundaries. However, the shape of the subgroups of IE are similar, and numerical difficulties are foreseen if more terms are considered. The functions IIA and IIB have been used with the three displacements formulation of the shell equations. Some essential or kinematic boundary conditions are not satisfied by some groups of these functions and in these cases Lagrange multipliers have been used.

4. Conclusions.

Shown above are several global numerical techniques that have been successively applied during the past for the analysis of shells. However, with the advent of the FEM and related numerical methods, global solutions had diminished use in shell analysis. Nevertheless, in the author's opinion, the simultaneous use of these solution techniques for the regular or smooth part of the shell structure and the application of a discretized model for the borders and the most irregular part of the shell can be a an efficient compromise.

Finally it is important to point out that shell analysis represents only a part of the more comprehensive task of the shell design. Problems of optimal shell shape finding, reinforcement of concrete shells and construction procedures must also be considered.

1. V. Z. Vlasov *General Theory of Shells and its Applications to Engineering* NASA. TTF-99 (1964).
2. A. Q. Samartin and J. Munro Dynamic *Analysis of Translational Shells* CSTR 67/2. Imperial College(London 1967).

3. S. A. Ambartsumyan *On the calculation of shallow shells* NACA. TM-1425 (English translation from Prikjanaya Matematika i Mekhankia, Vol **11** (1947).
4. J. Munro *The Linear analysis of thin shallow shells* Proc. Inst. Civil Engineers, Vol **19**. (1961).
5. A. L. Goldenweizer *Theory of elastic thin shells* Pergamon Press (1961).
6. D. A. Gunasekera *Numerical analysis of thin shells* PhD Thesis University of London (1967).
7. K. C. Michael and J. Munro *Approximating functions and indirect solutions of shell problems* Proc. of the IASS Conference (Mexico, 1967)
8. M. M. Filonenko-Boroditch *On a system of functions and its applications in the theory of elasticity* Prikjanaya Matematika i Mekhankia, Vol 10 (1963).